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RADIATIVE TRANSPORT IN AN ABSORBING PLANAR MEDIUM II. PREDICTIONS OF RADIATIVE SOURCE FUNCTIONS

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Abstract—An accurate solution of the source function for radiative heat transfer through a nonisothermal absorbing and emitting grey gas between heated plates is given in terms of tabulated functions. The prediction is based on an approximate formula originally proposed by Yamamoto for a grey gas atmosphere and is extended to allow for the presence of a uniform independent internal source of energy release. It is shown that knowledge of the Hopf function and the moments of the Chandrasekhar–Ambartsumian X and Y functions is sufficient to calculate solutions with a maximum fractional error of about 0·01 per cent. An alternative approach, based on Case's method of solution, is developed in the appendix and serves to confirm the functional forms of the approximate formulas. The results of this paper in conjunction with an earlier paper suffice to calculate the most important physical quantities with remarkable accuracy and simplicity.

NOMENCLATURE

$C(\xi_L)$, $D(\xi_L)$, coefficients introduced in equation (40);
$E_n(x)$, exponential integral function of order n defined by equation (3);
$H(\mu)$, Chandrasekhar's $H(\mu)$ function;
$\mathfrak{J}(\xi)$, radiative source function, equation (2);
k, volumetric absorption coefficient;
L, geometric thickness of plane layer;
$L(\xi_L)$, coefficient introduced in equation (26);
$M_1(\xi)$, $M_2(\xi_L)$, moments defined by equations (48) and (50);
$N(\xi, \xi_L)$, auxiliary function introduced in equation (33);
$q^*(\xi)$, Hopf function defined by equation (21);
q^+, q^-, half-range fluxes;
$Q(\xi_L)$, dimensionless flux defined by equation (15);
$Q_S(\xi_L)$, dimensionless flux defined by equation (16);
S, internal heat source per unit volume per unit time;
x, geometric depth in absorbing layer;
$X(\mu, \xi_L)$, Chandrasekhar–Ambartsumian X function;
$Y(\mu, \xi_L)$, Chandrasekhar–Ambartsumian Y function.

Greek symbols

$\alpha_n(\xi_L)$, nth moment of X function, see equation (10a);
$\beta_n(\xi_L)$, nth moment of Y function, see equation (10b);
γ, constant equal to twice the ratio of the second and first moments of $H(\mu)$, numerical value given following equations (14);

$\Theta(\xi, \xi_L), \Theta_S(\xi, \xi_L)$, universal functions introduced in equation (4), see also equations (5);

ξ, optical depth in absorbing layer;

$\Phi(\xi, \xi_L)$, resolvent kernel with one argument zero, see equation (7b);

$\Psi(\xi, \xi_L)$, auxiliary function defined by equation (7a).

INTRODUCTION

In a previous paper [1] we considered the prediction of radiant heat transfer through an absorbing and emitting grey gas held between heated opaque walls of infinite lateral extent. The paper had two principal objectives: first, to express in canonical form the basic equations that determine conditions through the medium for arbitrary wall temperatures, wall emissivities, total optical thickness, and for any magnitude of uniform internal energy production within the medium; second, to develop explicit predictions of radiant heat flux at the walls and of the discontinuities in temperature or the radiation source function at the boundaries. The latter objective was achieved by means of closed form solutions expressed in terms of moments of the Chandrasekhar–Ambartsumian X and Y functions. Calculation of the required predictions at the boundaries then became direct since tabulated values as well as analytic properties of the moment functions were available in the literature. Sobouti's [2] tables yield the necessary values correct to four decimal places.

The development of the basic equations resulted in the introduction of two universal functions Θ and Θ_S from which predictions for arbitrarily specified conditions could be generated (see, e.g. equation (46) in [1]). These functions, introduced explicitly in equation (4) below, are solutions of uncoupled integral equations. Among their many advantages are the facts that they treat separately the case of no internal energy release with unequal wall temperatures and uniform energy release with equal wall temperatures; also, they contain no explicit dependence on wall emissivities and need be calculated only for black wall conditions.

The present paper undertakes to extend the quantitative results given previously for the boundaries and to provide explicit predictions of the two universal functions through the medium by means of highly accurate, although approximate, formulas in terms of tabulated functions. The attainment of the objective is based on a formula first given by Yamamoto and a generalization developed in this paper to allow for the occurrence of uniform internal heat generation. The desired formulas [see equations (26) and (51)] either rival or exceed the accuracy in most published results of detailed numerical solutions, give proper predictions at the boundary of the medium, and the closed form relations may be manipulated analytically. The solutions are essentially the Eddington or diffusion approximations corrected by the addition of terms involving the Hopf function. An appendix indicates that independent derivations can be achieved by Case's method of normal mode expansion. The extension of similar analysis to more complicated radiation transfer problems remains to be considered.

The present results constitute a sequel to [1] and the formulas assembled in the two papers complete the list of the most important physical quantities. The terminology is unmodified with the exception of replacing $\sigma T^4/\pi$ by the radiative source function $\mathfrak{J}$. When internal heat generation is present, local thermodynamic equilibrium cannot necessarily be assumed, and consequently σT^4 should be replaced by $\pi\mathfrak{J}$ in the appropriate formulas of [1]. The emissive power introduced by some authors is equivalent to $\pi\mathfrak{J}$ in the present paper. The principal restrictions on the theory require that an appropriate averaging over the entire frequency spectrum is acceptable, that incident flux at the boundaries is isotropic, and that the internal heat generation is uniform.

References to earlier work in the radiation heat-transfer literature are given in [1] and are not repeated here.

BASIC EQUATIONS

We consider a unidimensional medium of finite width. The single coordinate x is measured normal to the boundary and traverses the region from left to right with boundaries at $x = 0$ and $x = L$. Let $k = k(x)$ be the volumetric absorption coefficient (assumed independent of frequency). The dimensionless optical distance ξ is, by definition,

$$\xi = \int_0^x k(x)\,\mathrm{d}x \qquad (1)$$

and when $x = L$, $\xi = \xi_L$ where ξ_L is called the optical thickness of the medium. Our major concern is the evaluation of the radiative source function $\mathfrak{I}(\xi)$ associated with a known rate of internal energy release S per unit volume within the medium and with a known level of isotropic radiation incident at each of the two boundaries. The source function for isotropic emission is defined such that $k\mathfrak{I}(\xi)$ is the amount of energy emitted per unit volume, per unit solid angle, and per unit time at the point ξ. Let q_{w1}^{+} and q_{w2}^{-} be the energy per unit time and area entering the medium from the left and right boundaries (or walls), respectively. The source function at an arbitrary position ξ is then, under equilibrium conditions, expressed as the sum of contributions from the incident energy flux, the internal sources, and from the integrated effects of all visible volume elements. For this particularly simple geometric configuration the source function satisfies the integral equation

$$\mathfrak{I}(\xi) = \frac{q_{w1}^{+}}{2\pi} E_2(\xi) + \frac{q_{w2}^{-}}{2\pi} E_2(\xi_L - \xi) + \frac{S}{4\pi k} + \frac{1}{2}\int_0^{\xi_L} \mathfrak{I}(\xi_1)\, E_1(|\xi - \xi_1|)\,\mathrm{d}\xi_1 \qquad (2)$$

where $E_n(\xi)$ is the nth exponential integral function

$$E_n(\xi) = \int_0^1 \exp(-\xi/\mu)\,\mu^{n-2}\,\mathrm{d}\mu = \int_1^\infty \exp(-\xi x)\,x^{-n}\,\mathrm{d}x. \qquad (3)$$

It should be remarked that in [1] local thermodynamic equilibrium was invoked, and consequently $\mathfrak{I}(\xi) = \sigma T^4/\pi$ where T is the local temperature and σ is the Stefan–Boltzmann constant. Here the more general form of the source function is retained.

Let $S/4\pi k$ be uniform throughout the medium. By virtue of the linearity of equation (2), the dependence of the source function on the parameters $S/4\pi k$, q_{w1}^{+}, and q_{w2}^{-} can be suppressed analytically through introduction of two dimensionless source functions or universal functions, Θ and Θ_S. To this end we introduce the transformation

$$\pi\mathfrak{I}(\xi) = q_{w2}^{-} + (q_{w1}^{+} - q_{w2}^{-})\Theta(\xi) + (S/k)\Theta_S(\xi). \qquad (4)$$

This relation has been previously applied to the study of an absorbing and emitting medium held within heated opaque walls, see equation (9) *et seq.* in [1].† The source function of equation (2) can then be calculated by means of equation (4) in terms of solutions of the two canonical integral equations (I-11).

$$\Theta(\xi, \xi_L) = \tfrac{1}{2}E_2(\xi) + \tfrac{1}{2}\int_0^{\xi_L} \Theta(\xi_1, \xi_L)\, E_1(|\xi - \xi_1|)\,\mathrm{d}\xi_1 \qquad (5a)$$

$$\Theta_S(\xi, \xi_L) = \tfrac{1}{4} + \tfrac{1}{2}\int_0^{\xi_L} \Theta_S(\xi_1, \xi_L)\, E_1(|\xi - \xi_1|)\,\mathrm{d}\xi_1. \qquad (5b)$$

Although in much of the literature of radiation theory the dependence on optical thickness ξ_L is not explicitly indicated, we will find it appropriate to emphasize this dependence and

† Henceforth, equations from [1] willl be referred to as, for example, (I-9).

equations (5) are written accordingly. Equation (4), which is written to conform with the symbolism of [1], follows conventional usage. Throughout this paper we have always $\Theta(\xi) = \Theta(\xi, \xi_L)$ and $\Theta_S(\xi) = \Theta_S(\xi, \xi_L)$.

Approximate but highly accurate expressions for the universal functions will be developed in the next section. Before proceeding to this task it is convenient to assemble a number of related formulas which will be required. These somewhat diverse results are consequences of the analysis of radiative transfer in plane-parallel media and are based on the methods given in the treatises of Chandrasekhar [4], Sobolev [5], and Kourganoff [6]. The auxiliary formulas are thus given without proofs.

From equations (I-40) and (I-41), the following precise relations apply

$$\Theta(\xi, \xi_L) = \frac{1}{2} - \frac{1}{2\Psi(\xi_L, \xi_L)}[\Psi(\xi, \xi_L) - \Psi(\xi_L - \xi, \xi_L)] \tag{6a}$$

and

$$\Theta_S(\xi, \xi_L) = \frac{\Psi(\xi_L, \xi_L)}{4}[\Psi(\xi, \xi_L) + \Psi(\xi_L - \xi, \xi_L) - \Psi(\xi_L, \xi_L)]. \tag{6b}$$

In this notation the function $\Psi(\xi, \xi_L)$ is, by definition,

$$\Psi(\xi, \xi_L) = 1 + \int_0^{\xi} \Phi(\xi_1, \xi_L)\, d\xi_1 \tag{7a}$$

where

$$\Phi(\xi, \xi_L) = L(0, \xi; \xi_L) \tag{7b}$$

and $L(\xi, \xi_1; \xi_L)$ is the symmetric resolvent kernel of the Fredholm equations (5) with kernel $E_1(|\xi - \xi_1|)$. It is obvious from equations (6) that $\Theta(\xi, \xi_L)$ is an odd function about the point $\xi = \xi_L/2$ and $\Theta(\xi_L/2, \xi_L) = \frac{1}{2}$, and that $\Theta_S(\xi, \xi_L)$ is an even function about $\xi = \xi_L/2$. The odd and even character of these functions will be of particular importance in the next section.

On the boundaries $\xi = 0, \xi_L$, the auxiliary function Ψ has the exact values

$$\Psi(0, \xi_L) = 1, \qquad \Psi(\xi_L, \xi_L) = 1/\beta_0(\xi_L) \tag{8}$$

and, consequently, the universal functions have the following precise values at the boundaries [1]:

$$\Theta(0, \xi_L) = \alpha_0(\xi_L)/2, \qquad \Theta(\xi_L, \xi_L) = \beta_0(\xi_L)/2 \tag{9a}$$

$$\Theta_S(0, \xi_L) = \Theta_S(\xi_L, \xi_L) = 1/[4\beta_0(\xi_L)] = (\tfrac{1}{8})/[1 - \Theta(0, \xi_L)] \tag{9b}$$

where α_0 and β_0 are zeroth moments of the Chandrasekhar–Ambartsumian X and Y functions [4]. The nth order moments are defined as

$$\alpha_n(\xi_L) = \int_0^1 X(\mu, \xi_L)\, \mu^n\, d\mu \tag{10a}$$

$$\beta_n(\xi_L) = \int_0^1 Y(\mu, \xi_L)\, \mu^n\, d\mu. \tag{10b}$$

The surface values (9) are related by the identity (I-32)

$$\alpha_0(\xi_L) + \beta_0(\xi_L) = 2. \tag{11}$$

Two additional useful identities [4] are

$$\xi_L[\alpha_1(\xi_L) + \beta_1(\xi_L)] + 2[\alpha_2(\xi_L) + \beta_2(\xi_L)] = \frac{4}{3\beta_0(\xi_L)} \tag{12}$$

and

$$\alpha_1(\xi_L) - \beta_1(\xi_L) = \xi_L\beta_0(\xi_L). \tag{13}$$

For a medium of large optical thickness, the following asymptotic moment relations [1, 7] apply

$$\beta_0(\xi_L) \sim \frac{2}{\sqrt{3}}\frac{1}{\gamma + \xi_L}, \qquad \xi_L \gg 1 \tag{14a}$$

$$\alpha_1(\xi_L) + \beta_1(\xi_L) \sim \frac{2}{\sqrt{3}} \tag{14b}$$

$$\alpha_2(\xi_L) + \beta_2(\xi_L) \sim \frac{\gamma}{\sqrt{3}} \tag{14c}$$

where $\gamma/2 = 0{\cdot}710446$ is the ratio of the second and first moments of Chandrasekhar's H function [6].

In [1], the flux integral (I-45a)

$$Q(\xi_L) = 1 - 2\int_0^{\xi_L} \Theta(\xi, \xi_L)\, E_2(\xi)\, d\xi = \beta_0(\xi_L)\,[\alpha_1(\xi_L) + \beta_1(\xi_L)] \quad (15)$$

is identified with the dimensionless flux of energy through a plane slab when no internal sources of energy release are present; the flux integral (I-45b)

$$Q_S(\xi_L) = 2\int_0^{\xi_L} \Theta_S(\xi, \xi_L)\, E_2(\xi)\, d\xi = \frac{\xi_L}{2} \quad (16)$$

establishes the proper value of the dimensionless flux associated with a uniform distribution of internal sources. One can also show that the integral of Ψ can be expressed as

$$\int_0^{\xi_L} \Psi(\xi, \xi_L)\, d\xi = \tfrac{3}{2}[\alpha_3(\xi_L) - \beta_3(\xi_L)] + \tfrac{3}{4}\xi_L[\alpha_2(\xi_L) - \beta_2(\xi_L)] + \frac{\xi_L^2}{8}[\alpha_1(\xi_L) - \beta_1(\xi_L)] + \frac{\xi_L}{2\beta_0(\xi_L)}. \quad (17)$$

The effect of an infinitesimal change in optical thickness on the flux integral (15) and the surface values (9) can be examined with the aid of the derivatives [4]

$$\frac{dQ(\xi_L)}{d\xi_L} = -\beta_0^2(\xi_L) \quad (18)$$

and

$$\frac{d}{d\xi_L}\frac{\alpha_0(\xi_L)}{2} = \frac{\beta_{-1}(\xi_L)\beta_0(\xi_L)}{4}. \quad (19)$$

Finally, we remark that auxiliary function (7), in the limit $\xi_L \to \infty$, becomes

$$\Psi(\xi, \infty) = \sqrt{3}\,[\xi + q^*(\xi)] \quad (20)$$

where $q^*(\xi)$ is the Hopf function that satisfies the integral equation [6]

$$q^*(\xi) = \tfrac{1}{2}E_3(\xi) + \tfrac{1}{2}\int_0^{\infty} q^*(\xi_1)\, E_1(|\xi - \xi_1|)\, d\xi_1. \quad (21)$$

The conventional notation for the Hopf function is q; however, this symbol was used in {[1], equations (I-6)} for total flux, and to avoid any possible confusion we denote the Hopf function by q^*. The solution of Milne's equation [6] for a semi-infinite atmosphere

$$B^*(\xi) = \tfrac{1}{2}\int_0^{\infty} B^*(\xi_1)\, E_1(|\xi - \xi_1|)\, d\xi_1 \quad (22a)$$

is given by

$$B^*(\xi) = \frac{\sqrt{3}}{4} F\Psi(\xi, \infty) = \tfrac{3}{4}F[\xi + q^*(\xi)] \quad (22b)$$

where πF is the uniform flux of radiant energy traversing the region in the negative x direction.

The Hopf function can be represented [6] by

$$q^*(\xi) = q^*(\infty) - \frac{1}{2\sqrt{3}}\int_0^1 \frac{\exp(-\xi/\mu)\, d\mu}{H(\mu)\, Z(\mu)} \quad (23)$$

where $H(\mu)$ is Chandrasekhar's H function [6] and

$$Z(\mu) = (1 - \mu \tanh^{-1}\mu)^2 + \tfrac{1}{4}\pi^2\mu^2. \quad (24)$$

The Hopf function is monotone increasing with the limited range

$$\frac{1}{\sqrt{3}} = q^*(0) \leqslant q^*(\xi) \leqslant q^*(\infty) = \frac{\gamma}{2} = 0{\cdot}710446 \quad (25a)$$

and q^* is within 0·09 per cent of its asymptotic value at an optical distance of 3. Thus for large optical thickness

$$q^*(\xi) \sim \frac{\gamma}{2} = 0{\cdot}710446. \quad (25b)$$

SOLUTIONS

The principal objective of this section is to consider results from different attacks on the fundamental integral equations and to show, through comparisons, that predictions of the source functions correct to within a small fraction of 1 per cent can be achieved in terms of tabulated functions. We note first the important contribution of Yamamoto [3] which was subsequently expressed more elegantly by King [8]. Yamamoto considered the physical problem of radiative transfer through a finite, plane-parallel, grey atmosphere heated solely from below by uniform isotropic radiation. (In meteorological applications the boundaries of the medium are horizontal rather than vertical.) The source function for this problem can be inferred directly from equation (4) by setting $S = 0$ because of the absence of independent internal sources and $q_{w2}^{-} = 0$ since there is no reflecting surface at the top of the atmosphere. Yamamoto based his proposed solution on a comparison of the approximate discrete ordinate solution of equation (5a) with the exact solution of the Milne problem. In the present notation, the functional form of the solution† is

$$\Theta(\xi, \xi_L) = \tfrac{1}{2} - \tfrac{3}{4}Q(\xi_L)\left(\xi - \frac{\xi_L}{2}\right) - \tfrac{3}{4}Q(\xi_L)\,L(\xi_L)\,[q^*(\xi) - q^*(\xi_L - \xi)] \qquad (26)$$

where $q^*(\xi)$ is the Hopf function defined by equation (21) and the coefficients Q and L remain to be determined as described below. Yamamoto identified $Q(\xi_L)$ as the dimensionless flux, defined by equation (15), which has been evaluated in terms of the moments of the Chandrasekhar–Ambartsumian X and Y functions. The necessary moments are tabulated, but a discussion of numerical results is deferred to the next section. It should be stressed that $Q(\xi_L)$ is equivalent to F/I_S in the notation of Yamamoto {[3], equation (21)} or F/I_g in the notation of King [8]. Thus $Q(\xi_L)$ of this paper does *not* conform with the Q appearing in the references just cited. Now let the coefficient $L(\xi_L)$ be fixed by requiring that solution (26) be exact at the boundaries. Thus, by equating solution (26) evaluated at $\xi = 0$ with the known surface value (9a)

$$\frac{\alpha_0(\xi_L)}{2} = \tfrac{1}{2} + \tfrac{3}{8}\xi_L Q(\xi_L) + \tfrac{3}{4}Q(\xi_L)\,L(\xi_L)\,[q^*(\xi_L) - q^*(0)] \qquad (27)$$

† In the Appendix we show that equation (26) can be deduced by Case's method of normal mode expansion.

one obtains, after solving for L and using identities (11) and (12),

$$L(\xi_L) = \frac{2/\sqrt{3}}{(\alpha_1 + \beta_1)}\,\frac{[(\sqrt{3}/2)(\alpha_2 + \beta_2) - 1/\sqrt{3}]}{[q^*(\xi_L) - 1/\sqrt{3}]}. \qquad (28)$$

In the above formula and those to follow we omit for simplicity the argument ξ_L of the moments α_n and β_n. Likewise, the argument ξ_L of Q and L will be omitted. The asymptotic behaviour of the above coefficients follows directly from equations (14) and (25b)

$$L \sim 1, \qquad \xi_L \gg 1 \qquad (29a)$$

$$Q \sim \frac{4}{3}\frac{1}{\gamma + \xi_L}. \qquad (29b)$$

Yamamoto's proposed solution (26) was based on intuitive considerations, and Sobolev [9] has, in fact, demonstrated that it is not exact. The derivative of $\Theta(\xi, \xi_L)$ becomes logarithmically infinite at both boundaries $\xi = 0$, ξ_L; Sobolev has shown that if the derivative of solution (26) has the required precise growth rate as a boundary is approached, then a logical consequence of Yamamoto's solution is

$$L = \frac{2}{\sqrt{3}}\frac{1}{\alpha_1 + \beta_1}. \qquad (30)$$

Equations (28) and (30) are consistent if and only if

$$q^* = \frac{\sqrt{3}}{2}(\alpha_2 + \beta_2). \qquad (31)$$

It is possible to establish, however, that equation (31) is incorrect. The tabular values of q^* in Kourganoff [6], for example, do not agree with evaluations of the right side of equation (31) by means of Sobouti's [2] tables of moments. An error of about 0·2 per cent occurs for small optical thicknesses. An independent check on the inaccuracy of equation (31) has also been given by Sobolev. One concludes, therefore, that the relation proposed by Yamamoto is not exact and that, as a consequence, an arbitrariness arises in the choice of L. We choose here the value of L provided by equation (28), that is, the value for which equation (26) predicts the exact value of $\Theta(\xi, \xi_L)$ at the boundaries.

Although equation (26) has been shown to be in error in its prediction of gradients near the boundaries, its incorrectness is virtually impossible to distinguish if one uses numerical tabulations with three-figure accuracy. This order of accuracy is typical of much of the numerical and graphical data given in the literature. In comparison with such data, therefore, equation (26) is equally acceptable, has analytic advantages, and yields exact magnitudes in the regions most difficult to study, that is, at the boundaries. We propose now to acknowledge equation (26) as a particularly accurate approximation of $\Theta(\xi, \xi_L)$ and to use its analytic simplicity, that is, its representation of a function of two variables as a sum of products of unidimensional functions, to achieve a comparable prediction of $\Theta_S(\xi, \xi_L)$. Since from equations (6) each of the universal functions is known once the auxiliary function $\Psi(\xi, \xi_L)$ is determined, it is appropriate to proceed by the intermediate step of developing a representation for Ψ. To this end, we note first, by a comparison of equations (6a) and (26), that $\Psi(\xi, \xi_L)$ satisfies the functional equation

$$\Psi(\xi, \xi_L) - \Psi(\xi_L - \xi, \xi_L) = \frac{3}{4}\frac{Q}{\beta_0} \times [\xi - (\xi_L - \xi)] + \frac{3}{2}\frac{QL}{\beta_0}[q^*(\xi) - q^*(\xi_L - \xi)]. \tag{32}$$

It follows that

$$\Psi(\xi, \xi_L) = \frac{3}{4}\frac{Q}{\beta_0}\xi + \frac{3}{2}\frac{QL}{\beta_0}q^*(\xi) + N(\xi, \xi_L) + B(\xi_L) \tag{33}$$

where $N(\xi, \xi_L)$ is a symmetric function about $\xi = \xi_L/2$, that is,

$$N(\xi, \xi_L) = N(\xi_L - \xi, \xi_L). \tag{34}$$

The additive term $B(\xi_L)$ was included in order to impose the convenient condition

$$N(0, \xi_L) = N(\xi_L, \xi_L) = 0. \tag{35}$$

When $\xi = 0$, the auxiliary function Ψ is unity and this condition is used to determine $B(\xi_L)$. Thus,

$$B(\xi_L) = 1 - \frac{\sqrt{3}}{2}\frac{QL}{\beta_0}$$

and Ψ can be rewritten as

$$\Psi(\xi, \xi_L) = 1 + \frac{3}{4}\frac{Q}{\beta_0}\xi + \frac{3}{2}\frac{QL}{\beta_0} \times [q^*(\xi) - 1/\sqrt{3}] + N(\xi, \xi_L). \tag{36}$$

One can easily verify that this expression for Ψ satisfies the right surface condition $\Psi(\xi_L, \xi_L) = 1/\beta_0$. It remains to deduce an expression for the symmetric function $N(\xi, \xi_L)$.

The relation between Ψ and Φ is given by equation (7a) and hence from equation (36)

$$\Phi(\xi, \xi_L) = \frac{\partial \Psi(\xi, \xi_L)}{\partial \xi} = \frac{3}{4}\frac{Q}{\beta_0} + \frac{3}{2}\frac{QL}{\beta_0}\frac{dq^*(\xi)}{d\xi} + \frac{\partial N(\xi, \xi_L)}{\partial \xi}. \tag{37}$$

For large optical thickness the coefficients are given by equations (14) and (29) and Φ becomes

$$\Phi(\xi, \xi_L) \sim \frac{\sqrt{3}}{2} + \sqrt{3}\frac{dq^*(\xi)}{d\xi} + \frac{\partial N(\xi, \xi_L)}{\partial \xi}. \tag{38}$$

Sobolev [10] has derived the following asymptotic formula for large optical thickness:

$$\Phi(\xi, \xi_L) \sim \frac{\sqrt{3}}{2} + \sqrt{3}\frac{dq^*(\xi)}{d\xi} + \frac{\sqrt{3}}{\gamma + \xi_L}\left(\frac{\xi_L}{2} - \xi\right) + \frac{\sqrt{3}}{\gamma + \xi_L}[q^*(\xi_L - \xi) - q^*(\xi)]. \tag{39}$$

A comparison of the last two equations suggests a valid approximation for $\partial N/\partial \xi$ throughout the full range of optical thickness, namely,

$$\frac{\partial N(\xi, \xi_L)}{\partial \xi} = C(\xi_L)\left(\frac{\xi_L}{2} - \xi\right) + D(\xi_L)[q^*(\xi_L - \xi) - q^*(\xi)] \tag{40}$$

where the coefficients C and D are to be determined for small optical thickness, and for large optical thickness they must have the asymptotic form

$$C(\xi_L) \sim D(\xi_L) \sim \frac{\sqrt{3}}{\gamma + \xi_L}. \quad (41)$$

The proposed form for Φ can now be written, from equations (37) and (40), as

$$\Phi(\xi, \xi_L) = \frac{3}{4}\frac{Q}{\beta_0} + \frac{3}{2}\frac{QL}{\beta_0}\frac{\mathrm{d}q^*(\xi)}{\mathrm{d}\xi} + C\left(\frac{\xi_L}{2} - \xi\right) + D[q^*(\xi_L - \xi) - q^*(\xi)]. \quad (42)$$

Evaluating this expression for Φ at the right surface $\xi = \xi_L$ by using the exact result $\Phi(\xi_L, \xi_L) = \beta_{-1}/2$, one obtains, after multiplying through by $\beta_0/2$,

$$\frac{\beta_{-1}\beta_0}{4} = \frac{3}{8}Q + \frac{3}{4}QL\frac{\mathrm{d}q^*(\xi_L)}{\mathrm{d}\xi_L} - C\frac{\beta_0}{4}\xi_L + D\frac{\beta_0}{2}[q^*(0) - q^*(\xi_L)]. \quad (43)$$

But by differentiating equation (27) and using equations (18) and (19), we obtain

$$\frac{\beta_{-1}\beta_0}{4} = \tfrac{3}{8}Q + \tfrac{3}{4}QL\frac{\mathrm{d}q^*(\xi_L)}{\mathrm{d}\xi_L} - \tfrac{3}{8}\beta_0^2\xi_L - \frac{3}{4}\frac{\mathrm{d}(QL)}{\mathrm{d}\xi_L}[q^*(0) - q^*(\xi_L)]. \quad (44)$$

These two expressions for $\beta_{-1}\beta_0$ are identical if we choose

$$C = \tfrac{3}{2}\beta_0 \quad (45)$$

$$D = -\frac{3}{2\beta_0}\frac{\mathrm{d}(QL)}{\mathrm{d}\xi_L}. \quad (46)$$

It follows from equations (14a, 18, 29) that these coefficients have the correct asymptotic form (41). The moment β_0 is a tabulated function, but the derivative of QL is expressed in terms of β_{-1} and the derivative of the Hopf function, neither of which is tabulated. Consequently, we resort to an alternative method to fix D.

After integrating equation (42), the auxiliary function Ψ is found to be

$$\Psi(\xi, \xi_L) = 1 + \int_0^{\xi} \Phi(\xi_1, \xi_L)\,\mathrm{d}\xi_1 = 1 + \frac{3}{4}\frac{Q}{\beta_0}\xi + \tfrac{3}{4}\beta_0(\xi\xi_L - \xi^2) + \frac{3}{2}\frac{QL}{\beta_0}[q^*(\xi) - q^*(0)] + D[M_1(\xi_L) - M_1(\xi_L - \xi) - M_1(\xi)] \quad (47)$$

where

$$M_1(\xi) \equiv \int_0^{\xi} q^*(\xi_1)\,\mathrm{d}\xi_1. \quad (48)$$

The remaining unspecified coefficient $D(\xi_L)$ is fixed by requiring that the integral of Ψ conform to the exact result (17). Thus by substituting equation (47) in (17) and solving for D, one finds, after using identities (12) and (13), that

$$D(\xi_L) = \frac{\frac{3}{2}(\alpha_3 - \beta_3) + \xi_L\left(\frac{\sqrt{3}}{2}\frac{QL}{\beta_0} + \frac{3}{2}\alpha_2 - 1\right) - \frac{3}{2}\frac{QL}{\beta_0}M_1(\xi_L)}{\xi_L M_1(\xi_L) - 2M_2(\xi_L)} \quad (49)$$

where

$$M_2(\xi_L) \equiv \int_0^{\xi_L} M_1(\xi)\,\mathrm{d}\xi = \int_0^{\xi_L}\mathrm{d}\xi\int_0^{\xi} q^*(\xi_1)\,\mathrm{d}\xi_1. \quad (50)$$

The goal of representing $\Theta_S(\xi, \xi_L)$ as a sum of products of unidimensional functions is now achieved by simply substituting equation (47) in (6b) to obtain

$$\Theta_S(\xi, \xi_L) = \frac{1}{4\beta_0} + \tfrac{3}{8}(\xi\xi_L - \xi^2) + \frac{3}{8}\frac{QL}{\beta_0^2}[q^*(\xi_L - \xi) + q^*(\xi) - q^*(\xi_L) - q^*(0)] + \frac{D}{2\beta_0}[M_1(\xi_L) - M_1(\xi_L - \xi) - M_1(\xi)]. \quad (51)$$

In obtaining this form of the solution, we have used identity (12) and the definition (28) of L.

the following formula of Huang [11] which was derived by variational methods:

$$q^*(\xi) = A_1 + \sum_{j=2}^{5} A_j E_j(\xi) \quad (52)$$

where

$A_1 = 0{\cdot}71044067 \qquad A_4 = -0{\cdot}61868635$

$A_2 = -0{\cdot}27894936 \qquad A_5 = 0{\cdot}35260116.$

$A_3 = 0{\cdot}52805161$

The exponential integral functions are tabulated by several authors, see, for example [6], p. 266, and formulas suitable for computing machine calculations are given by Abramowitz and Stegun ([12], p. 231). Table 1 lists five-place values of $q^*(\xi)$ computed for the Huang formula (52) for $\xi \geqslant 0{\cdot}01$.

Table 1. The Hopf function $q^*(\xi)$ and its integral $M_1(\xi)$

ξ	$q^*(\xi)$	$M_1(\xi)$	ξ	$q^*(\xi)$	$M_1(\xi)$	ξ	$q^*(\xi)$	$M_1(\xi)$
0	0·57735	0	0·60	0·68580	0·39337	2·20	0·70855	1·51677
0·01	0·58824	0·00584	0·65	0·68808	0·42772	2·30	0·70880	1·58764
0·02	0·59538	0·01176	0·70	0·69010	0·46217	2·40	0·70901	1·65853
0·03	0·60123	0·01774	0·75	0·69191	0·49672	2·50	0·70919	1·72944
0·04	0·60627	0·02378	0·80	0·69353	0·53136	2·60	0·70935	1·80037
0·05	0·61074	0·02986	0·85	0·69498	0·56607	2·70	0·70949	1·87131
0·06	0·61478	0·03599	0·90	0·69629	0·60085	2·80	0·70961	1·94226
0·07	0·61846	0·04216	0·95	0·69747	0·63570	2·90	0·70972	2·01323
0·08	0·62185	0·04836	1·00	0·69854	0·67060	3·00	0·70981	2·08421
0·09	0·62499	0·05459	1·10	0·70038	0·74055	3·10	0·70989	2·15519
0·10	0·62792	0·06086	1·20	0·70191	0·81067	3·20	0·70995	2·22618
0·15	0·64014	0·09257	1·30	0·70318	0·88092	3·30	0·71001	2·29718
0·20	0·64956	0·12483	1·40	0·70424	0·95129	3·40	0·71007	2·36819
0·25	0·65713	0·15750	1·50	0·70513	1·02176	3·50	0·71011	2·43920
0·30	0·66337	0·19052	1·60	0·70589	1·09232	3·60	0·71015	2·51021
0·35	0·66862	0·22382	1·70	0·70652	1·16294	3·70	0·71019	2·58123
0·40	0·67309	0·25737	1·80	0·70706	1·23362	3·80	0·71022	2·65225
0·45	0·67694	0·29112	1·90	0·70753	1·30435	3·90	0·71024	2·72327
0·50	0·68029	0·32505	2·00	0·70792	1·37512	4·00	0·71027	2·79429
0·55	0·68322	0·35914	2·10	0·70826	1·44593			

NUMERICAL CALCULATIONS

In the previous section, solutions of the universal functions $\Theta(\xi, \xi_L)$ and $\Theta_S(\xi, \xi_L)$ were expressed by equations (26) and (51) in terms of the unidimensional functions $q^*(\xi)$, $M_1(\xi)$, $Q(\xi_L)$, $L(\xi_L)$, $D(\xi_L)$, and $\beta_0(\xi_L)$. It remains to compute and tabulate these requisite functions. The task is direct once accurate values of the Hopf function and the basic moment functions are known.

Six-place tabular values of the Hopf function are given by Kourganoff ([6], p. 138). For computational purposes, a convenient representation is given to a high degree of accuracy by

The integral of the Hopf function (48) can also be evaluated by using the Huang formula (52):

$$M_1(\xi) = \int_0^{\xi} q^*(\xi_1)\,\mathrm{d}\xi_1 = A_1\xi + \sum_{j=2}^{5} A_j\left[\frac{1}{j} - E_{j+1}(\xi)\right]. \quad (53)$$

Likewise,

$$M_2(\xi) = \int_0^{\xi} M_1(\xi_1)\, d\xi_1 = A_1 \frac{\xi^2}{2} + \sum_{j=2}^{5} \times A_j \left[\frac{\xi}{j} + E_{j+2}(\xi) - \frac{1}{j+1} \right]. \quad (54)$$

Table 1 also includes tabular values of $M_1(\xi)$. From equation (23), the asymptotic integral of q^* is

$$M_1(\xi) \sim \frac{\gamma}{2}\xi + \frac{\gamma^2}{8} - \frac{3}{10} \quad (55a)$$

since

$$\frac{1}{2\sqrt{3}} \int_0^1 \frac{\mu\, d\mu}{H(\mu)\, Z(\mu)} = \frac{3}{10} - \frac{\gamma^2}{8}. \quad (55b)$$

The authors are not aware of a simple proof of identity (55b); the result given was obtained indirectly from analysis based on Case's method.

A table of the moment functions $\alpha_n(\xi_L)$ and $\beta_n(\xi_L)$ given by Sobouti [2] was used in our earlier paper [1]. Recently, a new and more accurate table of the Chandrasekhar–Ambartsumian X and Y functions has been computed by Carlstedt and Mullikin [13] (note that tabulated values for $\omega = 1$ are required in the present application). This latter table was used to compute the moments necessary for the present work. Table 2 includes numerical values for optical thicknesses up to 3·5 of $\beta_0(\xi_L)$ and the coefficients $Q(\xi_L)$ and $L(\xi_L)$ defined by equations (15) and (28). The asymptotic formulas at the bottom of the table may be used for larger values of optical thickness. An independent check on the values of Q is afforded by calculations of Mingle [14]. The flux $Q(\xi_L)$ is equivalent to the total transmission function for a grey slab computed by Mingle, and the tabulation of $Q(\xi_L)$ in Table 2 agrees with his numerical work to within one unit in the fifth decimal place. The coefficient $D(\xi_L)$ as defined by equation (49) also appears in Table 2. Some loss in the number of significant figures occurs in the calculation of D for small thickness. However, four-place accuracy can still be retained in the evaluation of Ψ and/or Θ_S because D is the coefficient of a quantity that is small when the optical thickness is small.

Table 2. The coefficients β_0, Q, L and D

ξ_L	$\beta_0(\xi_L)$	$Q(\xi_L)$	$L(\xi_L)$	$D(\xi_L)$
0·20	0·77713	0·84918	1·0383	1·46
0·40	0·66680	0·74585	1·0218	1·123
0·60	0·58966	0·66730	1·0138	0·950
0·80	0·53079	0·60473	1·0092	0·835
1·00	0·48370	0·55340	1·0063	0·750
1·20	0·44487	0·51037	1·0045	0·684
1·40	0·41215	0·47370	1·0032	0·629
1·60	0·38411	0·44204	1·0023	0·584
1·80	0·35977	0·41441	1·0017	0·545
2·00	0·33842	0·39006	1·0013	0·512
2·20	0·31951	0·36843	1·0009	0·482
2·40	0·30263	0·34909	1·0007	0·456
2·60	0·28748	0·33169	1·0005	0·433
2·80	0·27378	0·31595	1·0004	0·412
3·00	0·26135	0·30164	1·0003	0·3928
3·50	0·23472	0·27097	1·0002	0·3524
$\xi_L \gg 1$	$\frac{2/\sqrt{3}}{\gamma + \xi_L}$	$\frac{\frac{4}{3}}{\gamma + \xi_L}$	1	$\frac{\sqrt{3}}{\gamma + \xi_L}$

$\gamma = 1{\cdot}42089$

As a sample calculation, Table 3 lists numerical values of the universal function Θ and Θ_S obtained using formulas (26) and (51). Values are given only over the half range of ξ since the functions are known to be, respectively, odd and even. A comparison of these results with independent numerical solutions of the basic integral equations (5) computed by the authors [1] indicates that the maximum fractional error of the approximate predictions will be less than 0·01 per cent. In spite of the considerable literature regarding the problem of thermal radiative transfer, no numerical solutions of comparable accuracy have been found. An additional accuracy check was made by substituting solutions (26) and (51) into the flux integrals (15) and (16) and evaluating the integrals by numerical quadrature. The results were found to agree with the exact values $\beta_0(\alpha_1 + \beta_1)$ and $\xi_L/2$ to within $\pm 0{\cdot}00003$.

Table 3. Universal functions Θ and Θ_S

ξ/ξ_L	$\xi_L = 0{\cdot}2$ $\Theta(\xi,\xi_L)$	$\Theta_S(\xi,\xi_L)$	$\xi_L = 1{\cdot}0$ $\Theta(\xi,\xi_L)$	$\Theta_S(\xi,\xi_L)$	$\xi_L = 2{\cdot}0$ $\Theta(\xi,\xi_L)$	$\Theta_S(\xi,\xi_L)$
0	0·6114	0·3217	0·7581	0·5168	0·8308	0·7387
0·05	0·5967	0·3280	0·7230	0·5674	0·7866	0·8816
0·10	0·5845	0·3321	0·6946	0·6006	0·7509	0·9776
0·15	0·5730	0·3353	0·6682	0·6268	0·7174	1·0546
0·20	0·5620	0·3378	0·6429	0·6480	0·6851	1·1178
0·25	0·5513	0·3399	0·6183	0·6652	0·6535	1·1695
0·30	0·5408	0·3415	0·5942	0·6787	0·6224	1·2107
0·35	0·5305	0·3427	0·5704	0·6891	0·5916	1·2423
0·40	0·5203	0·3435	0·5468	0·6963	0·5610	1·2646
0·45	0·5101	0·3440	0·5234	0·7007	0·5305	1·2779
0·50	0·5000	0·3442	0·5000	0·7021	0·5000	1·2824

If equations (26) and (51) are evaluated for $\xi_L \gg 1$ and ξ distant from the boundaries by using formulas (14a, 25b, 29, 41, 55a), the following precise asymptotic forms are obtained

$$\Theta(\xi,\xi_L) \sim \frac{1}{2} - \frac{1}{\gamma + \xi_L}\left(\xi - \frac{\xi_L}{2}\right) \tag{56}$$

$$\Theta_S(\xi,\xi_L) \sim \tfrac{3}{8}(\xi\xi_L - \xi^2) + \tfrac{3}{16}\gamma\xi_L + \tfrac{3}{32}\gamma^2 + \tfrac{9}{40} \tag{57}$$

which aside from some numerical constants correspond to the Eddington approximations given by (I-64) and (I-69). Figure 1 shows the various contributions to $\Theta_S(\xi,\xi_L) - 1/(4\beta_0)$ given by the terms in formula (51) for an optical thickness of 2. It is clear that the parabolic Eddington approximation can only represent the solution near the center of the slab and then only if the vertex of the parabola is nearly coincident with the exact value of $\Theta(\xi_L/2,\xi_L)$. Both Θ and Θ_S are nonanalytic at the boundaries $\xi = 0$, ξ_L and, in fact, their derivatives behave as $E_1(\xi)$ as $\xi \to 0$ or $E_1(\xi_L - \xi)$ as $\xi \to \xi_L$. Clearly, this behaviour near the boundaries of a slab is not exhibited by the Eddington or diffusion approximations.

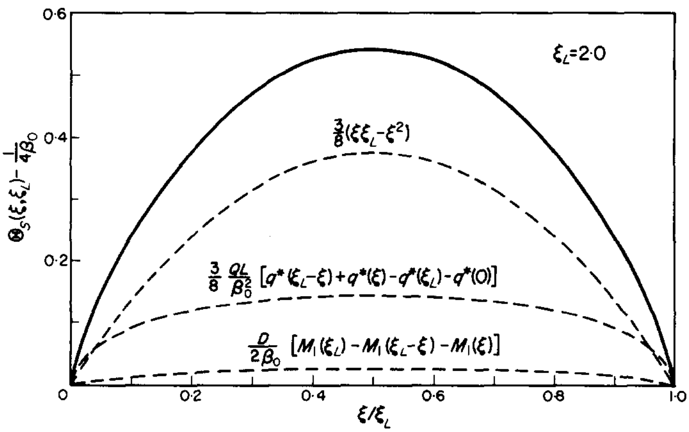

FIG. 1. The dashed curves represent the separate contributions to $[\Theta_S(\xi,\xi_L) - 1/(4\beta_0)]$ (solid curve) as given by the terms in equation (51).

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APPENDIX

The purpose of this appendix is to show that the functional form of the Yamamoto solution (26) is equivalent to the zeroth-order approximation obtained by Case's method [15] of normal mode expansion. The restrictions previously imposed on the properties of the medium, that is, grey gas, isotropic emission, etc., also apply to the following discussion.

We digress for a moment and recall that specific intensity for a plane-parallel slab satisfies the following equation of transfer [4]

$$\mu \frac{\mathrm{d}I(\xi, \mu)}{\mathrm{d}\xi} = -I(\xi, \mu) + \mathfrak{I}(\xi) \tag{A1}$$

where $\mu = \cos\theta$ and θ is the angle between the direction of increasing optical depth and a given direction. In the absence of independent internal sources, the source function $\mathfrak{I}(\xi)$ of a conservative radiation field is equal to the mean intensity $\bar{I}$ defined by

$$\bar{I} = \tfrac{1}{2} \int_{-1}^{1} I(\xi, \mu)\,\mathrm{d}\mu. \tag{A2}$$

Half-range intensities are introduced as

$$\left.\begin{aligned} I(\xi, \mu) &= I^{+}(\xi, \mu), & 0 \leqslant \mu \leqslant 1 \\ I(\xi, \mu) &= I^{-}(\xi, \mu), & -1 \leqslant \mu < 0 \end{aligned}\right\} \tag{A3}$$

and the half-range fluxes are defined by

$$q^{+}(\xi) = 2\pi \int_{0}^{1} \mu I^{+}(\xi, \mu)\,\mathrm{d}\mu \tag{A4a}$$

$$q^{-}(\xi) = 2\pi \int_{0}^{-1} \mu I^{-}(\xi, \mu)\,\mathrm{d}\mu \tag{A4b}$$

where the net flux is

$$q(\xi) = q^{+}(\xi) - q^{-}(\xi) = 2\pi \int_{-1}^{1} \mu I(\xi, \mu)\,\mathrm{d}\mu. \tag{A5}$$

Since our interest here is confined to the universal function $\Theta(\xi, \xi_L)$, we assume that the slab is irradiated by uniform isotropic radiation of unit magnitude at the left-hand boundary ($\xi = 0$) and that there is an absence of incident radiation at the right-hand boundary ($\xi = \xi_L$). The boundary conditions appropriate to the calculation of $\Theta(\xi, \xi_L)$ are thus

$$I^{+}(0, \mu) = 1, \qquad I^{-}(\xi_L, \mu) = 0 \tag{A6}$$

and from equations (A4), $q^{+}(0) = q_{w1}^{+} = \pi$ and $q^{-}(\xi_L) = q_{w2}^{-} = 0$. Consequently, the source function (4) is

$$\mathfrak{I}(\xi) = \Theta(\xi, \xi_L) = \tfrac{1}{2} \int_{-1}^{1} I(\xi, \mu)\,\mathrm{d}\mu. \quad \text{(A7)}$$

The specific intensity at any point is given by a formal solution of the equation of transfer (A1). For example, the variation with angle of the intensity emanating from the right-hand surface is

$$I^+(\xi_L, \mu) = \exp(-\xi_L/\mu) + \int_0^{\xi_L} \Theta(\xi, \xi_L) \exp[-(\xi_L - \xi)/\mu]\,\mathrm{d}\xi/\mu \quad \text{(A8a)}$$

$$= \frac{\beta_0(\xi_L)}{2}[X(\mu, \xi_L) + Y(\mu, \xi_L)]. \quad \text{(A8b)}$$

where the X and Y functions are tabulated by Sobouti [2] and Carlstedt and Mullikin [13]. The emergent intensity (A8b) was derived by King [16] by an invariant approach and was obtained here by utilizing equations (A8a), (6a), (I-30), and (7a). The flux at the right-hand surface is from (A8b) and (A4a)

$$\frac{q(\xi_L)}{\pi} = \frac{q^+(\xi_L)}{\pi} = \beta_0(\xi_L)[\alpha_1(\xi_L) + \beta_1(\xi_L)] \quad \text{(A9)}$$

which is identical to the dimensionless flux integral (15). It is important to recall that the total flux is constant across the slab in the absence of internal sources.

The Case method [15] for the present problem consists of an expansion of the intensity $I(\xi, \mu)$ in terms of the eigenfunctions of the transfer equation (A1) (with $\mathfrak{I}(\xi)$ replaced by (A2)). The continuum eigenfunctions are

$$\varphi_\nu(\mu) \exp(-\xi/\nu)$$

where

$$\varphi_\nu(\mu) = \frac{1}{2} P \frac{\nu}{\nu - \mu} + \lambda(\nu)\,\delta(\mu - \nu) \quad \text{(A10a)}$$

with

$$\lambda(\nu) = 1 - \nu \tanh^{-1} \nu \quad \text{(A10b)}$$

and the index ν is real and lies between -1 and 1. The P signifies that an integral with the factor $1/(\nu - \mu)$ is to be evaluated as a Cauchy principal value and $\delta(\mu - \nu)$ is the Dirac delta function. For radiative transfer in a grey gas ($c = 1$ in neutron transport literature) the two linearly independent discrete solutions [17] are a constant and a constant times $(\xi - \mu)$. Consequently, the specific intensity can be expanded as

$$I(\xi, \mu) = a + b(\xi - \mu) + \int_{-1}^{1} A(\nu)\,\varphi_\nu(\mu) \exp(-\xi/\nu)\,\mathrm{d}\nu \quad \text{(A11)}$$

where a, b, and $A(\nu)$ are arbitrary expansion coefficients which are fixed by requiring that boundary conditions (A6) be satisfied. The necessary normalization and orthogonality relations are given by Kuščer, McCormick and Summerfield [18]. Omitting the details of the analysis outlined by Ferziger and Simmons [19], one finds that the expansion coefficients satisfy

$$a = \tfrac{1}{2} - b\frac{\xi_L}{2} \quad \text{(A12)}$$

$$b = \frac{-1}{\gamma + \xi_L}\left[1 + \sqrt{3}\int_0^1 \frac{A(\nu)\exp(-\xi_L/\nu)\nu\,\mathrm{d}\nu}{H(\nu)}\right] \quad \text{(A13a)}$$

and

$$A(\nu) = \frac{-b}{\sqrt{3}\,H(\nu)\,Z(\nu)} + \frac{1}{2H(\nu)\,Z(\nu)}\int_0^1 \frac{A(\nu')\exp(-\xi_L/\nu')\nu'\,\mathrm{d}\nu'}{H(\nu')(\nu + \nu')}, \quad \nu > 0 \quad \text{(A13b)}$$

$$A(-\nu) = -A(\nu)\exp(-\xi_L/\nu) \quad \text{(A13c)}$$

where $H(\nu)$ is Chandrasekhar's H function [4], $Z(\nu)$ is defined by equation (24), and the numerical value of γ is given by equation (25a). The function $X(-\nu)$ appearing in the paper by Ferziger and Simmons [19] is equal to $\sqrt{3}/H(\nu)$ in the present notation.

Equations (A13) form a pair of coupled equations to be solved for the coefficients b and $A(\nu)$. Equations (A7, A11, A12, A13c) together with the normalization condition

$$\int_{-1}^{1} \varphi_\nu(\mu)\,d\mu = 1 \qquad (A14)$$

can be combined to give an expression for the source function:

$$\mathfrak{J}(\xi) = \Theta(\xi, \xi_L) = \frac{1}{2} + b\left(\xi - \frac{\xi_L}{2}\right) + \frac{1}{2}\int_0^1 A(\nu)\exp(-\xi/\nu)\,d\nu - \frac{1}{2}\int_0^1 A(\nu)\exp[-(\xi_L - \xi)/\nu]\,d\nu. \qquad (A15)$$

Apart from a numerical factor, the coefficient b represents the radiative flux

$$\frac{q(\xi)}{\pi} = -\tfrac{4}{3}b. \qquad (A16)$$

This is proved by introducing the intensity expansion (A11) into the flux expression (A5) and using the known equality

$$\int_{-1}^{1} \mu\varphi_\nu(\mu)\,d\mu = 0.$$

Since flux is constant across the slab, equations (A9) and (A16) may be equated and therefore

$$b = -\tfrac{3}{4}\beta_0(\xi_L)[\alpha_1(\xi_L) + \beta_1(\xi_L)] = -\tfrac{3}{4}Q(\xi_L). \qquad (A17)$$

Thus, the variation of b with optical thickness is known [see Table 2 for a tabulation of $Q(\xi_L)$].

Equation (A13b) is a Fredholm integral equation for $A(\nu)$ which can be solved by the method of successive approximations. Substitution of the lowest order approximation

$$A^{(0)}(\nu) = \frac{-b}{\sqrt{3}\,H(\nu)\,Z(\nu)} \qquad (A18)$$

in equation (A15) yields, after noting identity (A17),

$$\Theta(\xi, \xi_L) \cong \tfrac{1}{2} - \tfrac{3}{4}Q(\xi_L)\left(\xi - \frac{\xi_L}{2}\right) - \frac{\sqrt{3}}{8}Q(\xi_L)\left[-\int_0^1 \frac{\exp(-\xi/\nu)\,d\nu}{H(\nu)\,Z(\nu)} + \int_0^1 \frac{\exp[-(\xi_L - \xi)/\nu]\,d\nu}{H(\nu)\,Z(\nu)}\right]. \qquad (A19)$$

But by using the Hopf function as given by equation (23), equation (A19) can be rewritten as

$$\Theta(\xi, \xi_L) \cong \tfrac{1}{2} - \tfrac{3}{4}Q(\xi_L)\left(\xi - \frac{\xi_L}{2}\right) - \tfrac{3}{4}Q(\xi_L)[q^*(\xi) - q^*(\xi_L - \xi)]. \qquad (A20)$$

Apart from the absence of the last term multiplicative factor $L(\xi_L)$ which is of order one (see Table 2), this solution is identical to the proposed solution (26). Since higher order terms are neglected, the solution is, of course, not exact and consequently the coefficient of $[q^*(\xi) - q^*(\xi_L - \xi)]$ is not unique. However, a more accurate solution can be anticipated if the coefficient is assumed to be an unspecified constant which is then fixed by some additional constraint. This, in fact, was the way in which the actual coefficient QL in equation (26) was determined.

The functional form of the Θ_S solution (51) can also be deduced by using Case's method. However, the analysis is less direct and will not be presented here.

Résumé—On donne à l'aide de fonctions tabulées une solution précise de l'expression fonctionnelle de la source pour le transport de chaleur par rayonnement à travers un gaz gris non isotherme absorbant et émetteur placé entre des plaques chauffées. La prévision est basée sur une formule approchée proposée originellement par Yamamoto pour une atmosphère gazeuse grise et on l'a étendue pour tenir compte de la présence d'une source interne de dégagement d'énergie indépendante et uniforme. On montre que la connaissance de la fonction de Hopf et des moments des fonctions X et Y de Chandrasekhar–Ambartsumian est suffisante pour calculer les solutions avec une erreur relative maximale de 0,01 pour cent. Une autre approximation basée sur la méthode de résolution de Case, est exposée dans l'appendice et sert à confirmer les expressions des formules approchées. Les résultats de cet article en liaison avec un article antérieur suffisent pour calculer les quantités physiques les plus importantes avec une précision et une simplicité remarquables.

Zusammenfassung—Eine genaue Lösung der Quellfunktion für den Strahlungswärmetransport durch ein nichtisothermes, absorbierendes und emittierendes graues Gas zwischen zwei beheizten Platten wird in Form von tabellierten Funktionen gegeben. Die Berechnung beruht auf einer ursprünglich von Yamamoto für eine graue Gasatmosphäre angegebene Näherungsgleichung und wird erweitert um eine gleichmässige unabhängige Energieabgabe zu erfassen. Es wird gezeigt, dass die Kenntnis der Hopf-Funktion und der Momente der Chandrasekhar–Ambartsumian X und Y Funktionen genügt, um die Lösungen mit einem maximalen Teilfehler von etwa 0,01% zu berechnen. Eine auf der Lösungsmethode von Case beruhende Alternativnäherung wird zur Bestätigung der Funktionalformen der Näherungsgleichungen im Anhang wiedergegeben. Die Ergebnisse dieser Arbeit in Verbindung mit einer früheren genügen, um die meisten wichtigsten physikalischen Grössen mit bemerkenswerter Genauigkeit und Einfachheit zu ermitteln.

Аннотация—В виде протабулированых функций приводятся точные решения функции источника для лучистого теплообмена, в неизотермической излучающе-поглощающей серой газовой среда, заключенной между нагретыми пластинками. Расчет основан на приближенной формуле Ямамато, модифицированной для учета независимого однородного источника испускаемой энергии. Показано, что для решения с максимальной погрешностью 0,01% достаточно знать функцию Хопфа и моменты функций X и $У$ Чандрасекхара и Амбарцумяна. В приложении излагается другой подход, основанный на методе Кейза, который служит для подтверждения функциональных форм приближенных формул. Результаты этой работы совместно с данными предыдущей статьи могуи применяться для расчета наиболее важных физических величин с высокой точностью и простотой.